Math 16a Review Topics

Monday, December 7, 2020 9:28 PM

- **Start Recording!!**
- Optimization
  - example: find the minimum area of a rectangle with total perimeter 4
  - determine the points on $y = x^2 + 1$ that are closest to $(0,2)$
  - **my general solution technique:**
    - (1) determine the function we want to optimize (mathematical modeling)
    - (2) determine any constraints (domain constraints or variable constraints)
    - reduce the problem to finding absolute extrema of a function.

$A(x,y) = xy$

$2x + 2y = 4$ Constraint!

Solve for x or y

$2y = 4 - 2x$

$y = 2 - x$

Substitute into $A(x,y) = x(2 - x) = 2x - x^2$

Want to optimize $2x - x^2$

minimize $2x - x^2 = A(x)$

$A'(x) = 2 - 2x = 0 \Rightarrow x = 1$

$0 \le x \le 2$

$0 < x < 2$

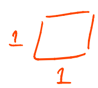
$A(0) = 0$

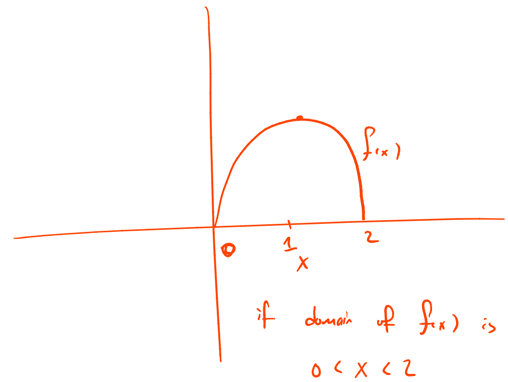
$A(2) = 0$

$A(1) = 2 - 1 = 1$

$A''(x) = -2$

max at $x = 1$

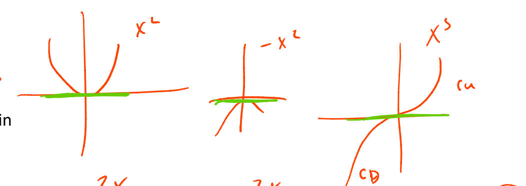




if domain of $f(x)$ is $0 < x < 2$

Then there is no minimum.

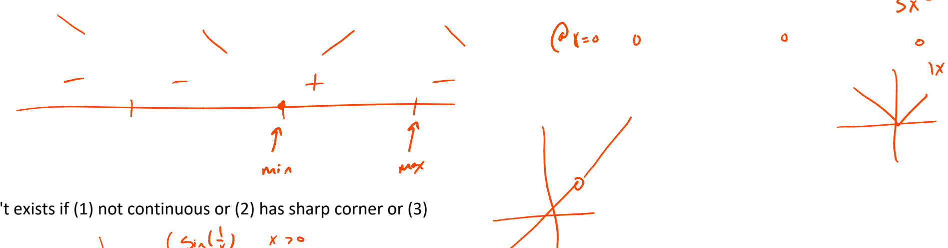
- Maximization/minimization
  - example: find absolute extrema of $f(x) = x^2$ on $[-1,2]$
  - **my general solution technique**
    - take derivative, find where it is zero, or doesn't exist, also check end points
    - check each of these points
      - second derivative test: if $f''(x_0) < 0 \Rightarrow$ max (concave down), if $f''(x_0) > 0 \Rightarrow$ min (concave up)
      - first derivative test (plug in points)



$2x$ $\quad -2x \quad 3x^2 \to 6x$

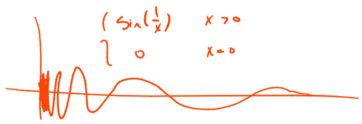
@ $x = 0$: 0 $\quad 0 \quad 0$

$|x|$



The derivative doesn't exists if (1) not continuous or (2) has sharp corner or (3) rapidly oscillating

$$\begin{cases} \sin(\frac{1}{x}) & x > 0 \\ 0 & x = 0 \end{cases}$$



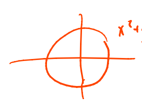
- related rates
  - example: the radius of a sphere is growing at 1 meter per second, how fast is the volume changing?
  - my general solution:
    - find a relationship between all the variables
    - find the derivative using implicit differentiation
- implicit differentiation
  - example: find $\frac{dy}{dx}$ for $y^2 + x^2 = 1$
  - solution
    - consider y as a function of x, rewrite as $y(x)$
    - take the derivative, remembering the chain rule
    - solve for $y'(x)$

$V(r) = \frac{4}{3}\pi r^3$

$\frac{d}{dt} V(r) = \frac{4}{3}\pi\, 3r^2(t) \frac{dr}{dt}$

$\frac{dr}{dt} = 1$

$\frac{d}{dt} V(r) = 4\pi r^2(t)$



$x^2 + y^2 = 1$

$y = \pm\sqrt{1 - x^2}$

$V(r) = \frac{4}{3}\pi r^3$

radius is growing

$r(t)$ is fn of time.

$V(r(t)) =$

$V = \frac{4}{3}\pi r^3(t)$

$\frac{d}{dt} V = \frac{d}{dt}(\quad)$

$y^2(x) + x^2 = 1$

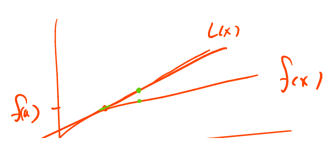
$\frac{d}{dx}(y^2(x) + x^2) = \frac{d}{dx}(y^2(x)) + \frac{d}{dx}(x^2) \quad \frac{d}{dx}(1)$

$= 2y(x)\, y'(x) + 2x \quad = 0$

$2y(x)\, y'(x) = -2x$

$y'(x) = \frac{-2x}{2y(x)} = \frac{-x}{y}$

- linear approximation
  - these problems are usually just **approximating** a function f, by a tangent line at a point
  - there are several equivalent formula.
    - $L(x) = f(a) + f'(a)(x - a)$



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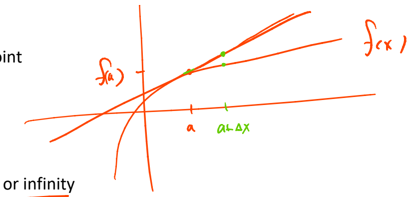
  - example approximate $(8.05)^{\frac{1}{3}}$



$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

- L'Hopital's rule
  - used to compute $\lim_{x\to a} \frac{f(x)}{g(x)}$ if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ **both** go to either zero or infinity
  - then this is just: $\lim_{x\to a} \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)}$
  - we can do this over and over again.
- Asymptotes
  - horizontal asymptotes are found by $\lim_{x\to\pm\infty} f(x)$
  - vertical asymptotes are found when $\lim_{x\to a} f(x) = \pm\infty$
  - oblique asymptotes are found when $f(x)$ looks like a slanted line if we zoom out
- Polynomial Division
  - go through an example
  - find the oblique asymptotes of $y = \frac{-3x^2+2}{x-1}$
- Tangent Lines to curves
  - same as linear approximation
- integrals
  - integrals of $\boxed{x^n, \frac{1}{x}, e^x, a^x, \cos(x), \sin(x), kf(x), f(x) \pm g(x)}$
- u-substitution
  - $\int \frac{3}{x\ln(x)} dx$
  - $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$ $u = x^2$
  - $\int x\sqrt{4-x}dx$
  - $\int \frac{x^2+4}{x+2}dx$

$\int \frac{3}{x\ln(x)} dx$

Let $u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

Solve for dx

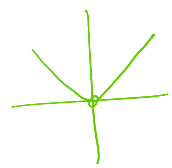
then $x\,du = dx$

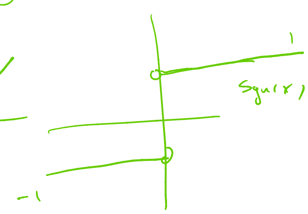
$\int \frac{3}{\cancel{x}u} \cancel{x}\,du$

$= 3\int \frac{1}{u} du = 3\ln(u)$

$= 3\ln(\ln(x))$

- derivative of absolute value
  - $\frac{d}{dx}|x| = \mathrm{sgn}(x)$
  - $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(abs(x)) = \frac{1}{abs(x)}abs'(x) = \frac{\mathrm{sgn}(x)}{|x|} = \frac{1}{x}$





- module 4
  - concavity
  - curve sketching
- numerical integration
  - Simpson's rule
    - $\int_a^b f(x)dx \approx \frac{b-a}{3n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$
  - Trapezoidal rule
    - $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$
  - midpoint rule
    - $\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(f(x_1^*) + f(x_2^*) + \cdots + f(x_6^*)\right)$
  - left and right also have the same formula as above.